Procedure to Reduce the Effects of Modal Truncation in Eigensolution Reanalysis

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In the context of sensitivity analysis, the influence of structural perturbations on the eigensolutions of a mechanical system is calculated through the use of a discrete model composed of the first normal modes of the base system. The method proposed in this study attempts to diminish the effects of modal truncation. It consists of modifying the structural perturbations before the application of the classical sensitivity relationships. The application of a simple system allowed testing of the efficiency of the procedure.

Introduction

THE goal of eigensensitivity analysis is to calculate variations in mode shapes and natural frequencies induced by modifying the terms in the mass and stiffness matrices of a structure.[1-3] Such an approach allows one to locate possible structural modifications to increase one or several resonance frequencies. It also has the advantage of not requiring a complete calculation of the new eigenvalues of perturbed systems and thereby avoiding excessive calculation time. It may be used to estimate the gradients of a cost function, bringing the modal characteristics into play in the framework of an optimal design procedure[4,5] or a correction of finite-element models based on vibration tests.[6,7]

With an eye to simplifying the analysis, we consider a structure to be undamped and conservative. By using the finite-element method, the normal modes may be approached in the following form:

$$KX_k = \omega_k^2 MX_k \tag{1}$$

The objective is to obtain, through the use of the simplest possible calculation, the solutions of the following perturbed eigenvalue problem:

$$(K + \Delta K)\bar{X}_k = \bar{\omega}_k^2(M + \Delta M)\bar{X}_k \tag{2}$$

According to the procedures used, variations ΔK and ΔM have a more or less weak norm with respect to the base problem matrices defined in Eq. (1). We notice that

$$\Delta K = \epsilon A \tag{3a}$$

$$\Delta M = \epsilon B \tag{3b}$$

The natural approach is to apply the perturbation method to the eigensolutions,

$$\bar{\omega}_k^2 = \bar{\lambda}_k = \omega_k^2 + \epsilon\alpha_{1k} + \epsilon^2\alpha_{2k} + \cdots \tag{4a}$$

$$\bar{X}_k = X_k + \epsilon R_{1k} + \epsilon^2 R_{2k} + \epsilon^3 R_{3k} + \cdots \tag{4b}$$

Received July 31, 1987; revision received Aug. 1, 1988.

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In practice, we shall limit ourselves to cases of moderate perturbations and to an expansion to the second order.

If the modifications apply to only a low number of nodes in the finite-element model, it is advantageous to introduce the corresponding flexibility matrix. Thus, Eq. (2) may be expressed in the following form:

$$S(\Delta\bar{\bar{K}} - \bar{\omega}_k^2\Delta\bar{\bar{M}})\{X_{ik}\} = 0 \tag{5a}$$

$$S = [S_{ij}] \tag{5b}$$

The flexibility matrix S and matrices $\Delta\bar{\bar{K}}$ and $\Delta\bar{\bar{M}}$ correspond to restrictions of matrices to the nodes concerned by the modification. Hence, S_{ij} is composed of certain terms of the total flexibility matrix $(K - \omega^2 M)^{-1}$, where ω is an arbitrary scalar frequency. These terms may be expressed with the help of N mode shapes,

$$S_{ij} = \sum_{k=1}^{N} \frac{X_{ik}X_{jk}}{-\omega^2 + \omega_k^2} \tag{6}$$

with $X_k = \{X_{ik}\}$.

The new natural frequencies $\bar{\omega}_k$ appear as values with roots in the determinant of the homogeneous system of Eq. (5), as

$$\det[S(\bar{\omega}_k)(\Delta\bar{\bar{K}} - \bar{\omega}_k^2\Delta\bar{\bar{M}})] = 0 \tag{7}$$

A nonlinear problem in the form of that of Weissemberger[8,9] is obtained and may be resolved by a Newton-Raphson procedure.[10,11] Inversely, the procedure proposed by Halliquist and Feng[12] allows one to ascertain from this equation the different choices of $\Delta\bar{\bar{K}}$ and $\Delta\bar{\bar{M}}$ resulting in imposed values of $\bar{\omega}_k$.

In agreement with the modal synthesis methods,[13,14] the new eigensolutions also may be obtained as solutions to the following problem:

$$(\Omega^2 + X^t\Delta KX)Q_k = \bar{\omega}_k^2(I + X^T\Delta MX)Q_k \tag{8}$$

with

$$\Omega^2 = [\omega_i^2\delta_{ij}] \tag{9a}$$

$$I = [\delta_{ij}] \tag{9b}$$

$$X = [X_{ij}] \tag{9c}$$

where δ_{ij} is the Kronecker delta and modal matrix X collects the displacements induced by the N first modes of the base structure. Nevertheless, the latter procedure does not allow one to link the new solutions analytically to those of the base problem of Eq. (1). To obtain direct relations, it is necessary to calculate the terms arising from the expansion given in Eqs. (4).

The procedure used by Ryland and Meirovitch[15] rests on a Cholesky decomposition of the mass matrix and a projection in the base of the eigenvectors. Thus, after having carried over Eqs. (4) into Eq. (2), while identifying the terms of the first order, they obtain[15]

$$\alpha_{1k} = X_k^T(A - \omega_k^2 B)X_k \tag{10}$$

$$R_{1k} = -\tfrac{1}{2}X_k^T B X_k X_k - \sum_{\substack{i=1\\ i\neq k}}^{N} \frac{X_k X_k^T}{\omega_k^2 - \omega_k^2}(-\omega_k^2 B + A)X_k \tag{11}$$

The terms of order zero are linked to the orthogonality conditions of the normal modes:

$$X_i^T M X_j = \delta_{ij} \tag{12}$$

$$X_i^T K X_j = \omega_i \omega_j \delta_{ij} \tag{13}$$

It is observed that, at order one, the natural frequency correction of mode k does not bring the other modes into play. To obtain a better estimation, it is necesary to use Eq. (4a) at order two:

$$\alpha_{2k} = -X_k^T(-\omega_k^2 B + A)\left[X_k X_k^T B + \sum_{\substack{i=1\\ i\neq k}}^{N} \frac{X_k X_k^T}{\omega_i^2 - \omega_k^2}(-\omega_k^2 B + A)X_k\right] \tag{14}$$

The use of flexibility matrices introduced in Eqs. (5) allows one also to obtain the terms of the expansion indicated in Eqs. (4). Indeed, by causing the variation of the following trivial identity:

$$(K - \omega_k^2 M)(K - \omega_k^2 M)^{-1} = I \tag{15}$$

one obtains

$$(\Delta\bar{\bar{K}} - \omega_k^2 \Delta\bar{\bar{M}})S + S^{-1}\Delta S = 0 \tag{16a}$$

that is,

$$\Delta S = -S(\Delta\bar{\bar{K}} - \omega_k^2 \Delta\bar{\bar{M}})S \tag{16b}$$

By making the two members of this equation explicit through the use of Eq. (6) and by identifying the polar decomposition coefficients, one arrives at the relations at order one obtained previously. Vanhonacker[16] used this method to obtain the sensitivity relationships in the case of damped structures. Thanks to this same procedure, he obtains second-order terms identical to those given in Eq. (14) in the undamped case. Brandon[17] obtained the same results by applying the perturbation method to complex modes.

Sensitivity analysis has developed considerably as is shown by a remarkable review of the current state-of-the-art carried out by Baldwin and Hutton.[18] Nonetheless, it seems that few studies have dealt with the influence of the modal truncation in the calculation of eigenvalue derivatives. In the case of the use of Eqs. (5), Hirai and Yoshimura[8] proposed the introduction of residual terms to reduce the influence of truncation in the expression of the flexibility matrix given in Eq. (6),

$$S_{ij} = \sum_{L=1}^{J} \omega^{2(L-1)} S_{Rij}^{(L)} + \sum_{k=1}^{N} \frac{X_{ik}X_{jk}}{-\omega^2 + \omega_k^2} \tag{17}$$

where J is the order of the residual flexibility.

In agreement with the work of Leung,[19-21] the residual terms are given by

$$S_{Rij}^{(L)} = G_{ij}^{(L)}(\omega = 0) - \sum_{k=1}^{N} \omega_k^{-2L} X_{ik} X_{jk} \tag{18a}$$

with

$$G^{(L)} = \left(\frac{\partial}{\partial\omega^2}\right)^{(L-1)} S \tag{18b}$$

In the context of modal synthesis, the use of residual characteristics has been proposed by numerous authors.[22-23] By limiting oneself to terms of the first order, the natural frequencies of the perturbed system are solutions of the following eigenvalue problem:

$$\begin{bmatrix} \Omega^2 + X_F^T K_R X_F & -X_F^T K_R \\ -K_R X_F & K_R + \Delta K \end{bmatrix} \begin{Bmatrix} Q \\ V \end{Bmatrix}_k = \bar{\omega}_k^2 \begin{bmatrix} I & 0 \\ 0 & \Delta M \end{bmatrix} \begin{Bmatrix} Q \\ V \end{Bmatrix}_k \tag{19}$$

$$K_R = S_R^{-1} \tag{20}$$

where displacement coordinates V correspond to the boundary nodes involved in the structural perturbation. Matrix X_F gathers the boundary displacements induced by the N modes retained.

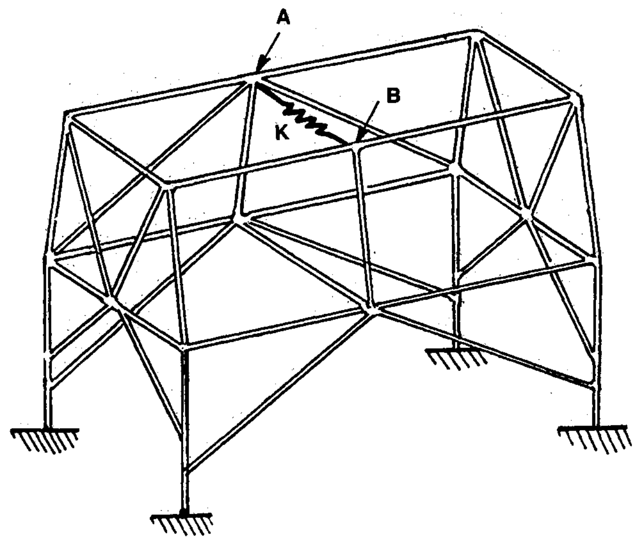

Fig. 1 Beam finite-element model.

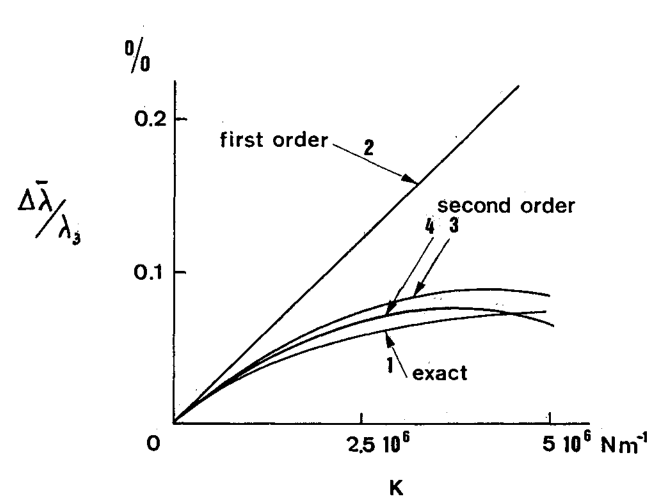

Fig. 2 Effect of stiffness K on the third natural frequency of the structure shown in Fig. 1.

In the case of the use of sensitivity methods based on a perturbation analysis defined in Eqs. (4), to our knowledge, no study of the influence of modal truncation has been made. This omission is certainly due to the fact that the term of the first order given in Eq. (10) does not bring out the coupling among the modes. In the case of the beam assembly shown in Fig. 1 and discretized through the use of 72 finite elements, the evolution of the third resonance frequency was calculated when a spring with an increasing stiffness was introduced between nodes A and B. In Fig. 2, curve 2 corresponds to the use of Eq. (10) and curves 3 and 4 to the calculation at the second order when the first 10 and 20 normal modes, respectively, are used. The modal truncature clearly influences the results obtained for the second order when the perturbation increases. In order to improve the results, it seems natural to use Eq. (17). Sensitivity Eqs. (11) and (14) then take on the following form:

$$R_{1k} = -\tfrac{1}{2}X_k^TBX_kX_k - \sum_{\substack{i=1\\ i\neq k}}^{N}\left(\sum_{L=1}^{J}\omega_k^{2(L-1)}S_R^{(L)} + \frac{X_kX_k^T}{\omega_i^2-\omega_k^2}\right) \times (-\omega_k^2B + A)X_k \tag{21a}$$

$$\alpha_{2k} = -X_k^T(-\omega_k^2B + A)\left[X_kX_k^TB + \sum_{\substack{i=1\\ i\neq k}}^{N} \times \left(\sum_{L=1}^{J}\omega_k^{2(L-1)}S_R^{(L)} + \frac{X_kX_k^T}{\omega_i^2-\omega_k^2}\right)(-\omega_k^2B + A)X_k\right] \tag{21b}$$

Although these expressions allow us to reduce errors during the calculation of the terms introduced in Eqs. (4), it is interesting to seek a procedure that generally improves the convergence of Eqs. (4) when a modal truncation is introduced. The method we propose is based on a very simple idea that consists of applying classical relationships, Eqs. (10), (11), and (14), not to the real system but to a modal model, Eq. (19), that takes the modal truncation into account in the form of a residual stiffness matrix K_R.

Presentation of the Method

An important application of sensitivity analysis consists of studying the influence of a local modification of a structure's stiffness. If one considers that the nodes concerned by the modification correspond to the displacement of the boundary occurring in the discrete model defined in Eq. (19), the problem can be described by the schematic graph shown in Fig. 3a. Coordinates U_i correspond to the modal model [Eq. (8)] and are directly linked to modal coordinate vector Q as

$$\{U_i\} = X_FQ \tag{22}$$

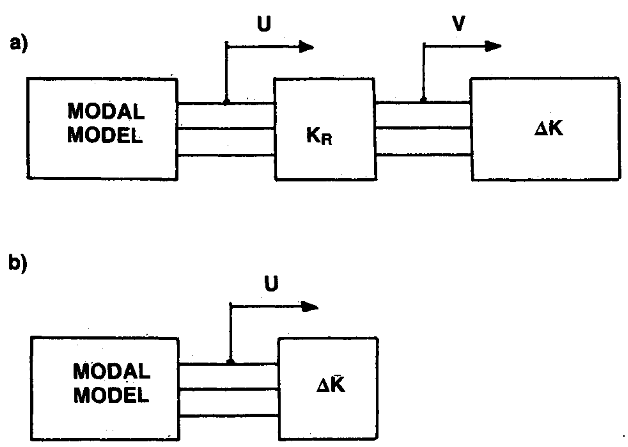

Fig. 3 Schematic diagrams of the stiffness perturbation.

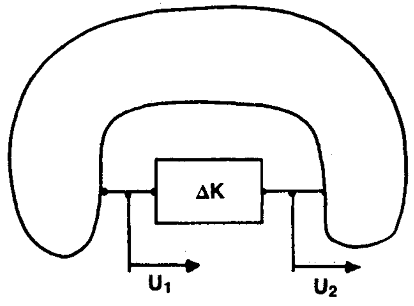

Fig. 4 Introduction of a stiffener between 2*n* degrees of freedom.

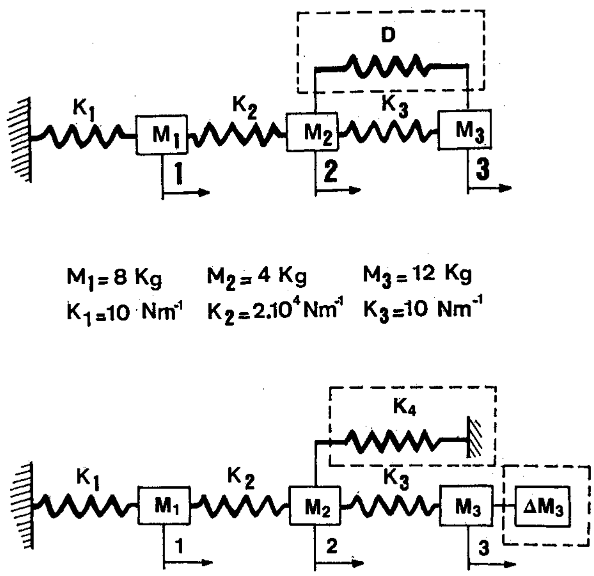

Fig. 5 Three-degrees-of-freedom system with stiffness and mass perturbation.

Displacement coordinates V_i are linked to coupling forces F_i exercised by the structure on the stiffener,

$$\{F_i\} = \Delta K\{V_i\} \tag{23}$$

where ΔK is the stiffnes matrix perturbation. Displacement vectors U and V verify the following relationship:

$$F = K_R(U - V) \tag{24}$$

The residual stiffness matrix K_R takes into account the influence of the nonretained modes. This matrix is obtained thanks to static flexibility S_T composed of the terms of the matrix K^{-1} associated with the boundary nodes, as

$$K_R^{-1} = S_T - X_F\Omega^{-2}X_F^T = S_R \tag{25}$$

If the structure is free standing, matrix S_T is calculated by eliminating the participation of the rigid-body modes through the use of the procedure described by Rubin.[24] If the displacement vector V in the system of Eqs. (23) and (24) is eliminated, one obtains

$$(I + \Delta KS_R)F = \Delta KU \tag{26}$$

Thus, everything takes place as though the modal model of Eq. (8) were linked to a new stiffness matrix perturbation $\Delta\tilde{K}$ of the following form:

$$\Delta\tilde{K} = (I + \Delta KS_R)^{-1}\Delta K \tag{27}$$

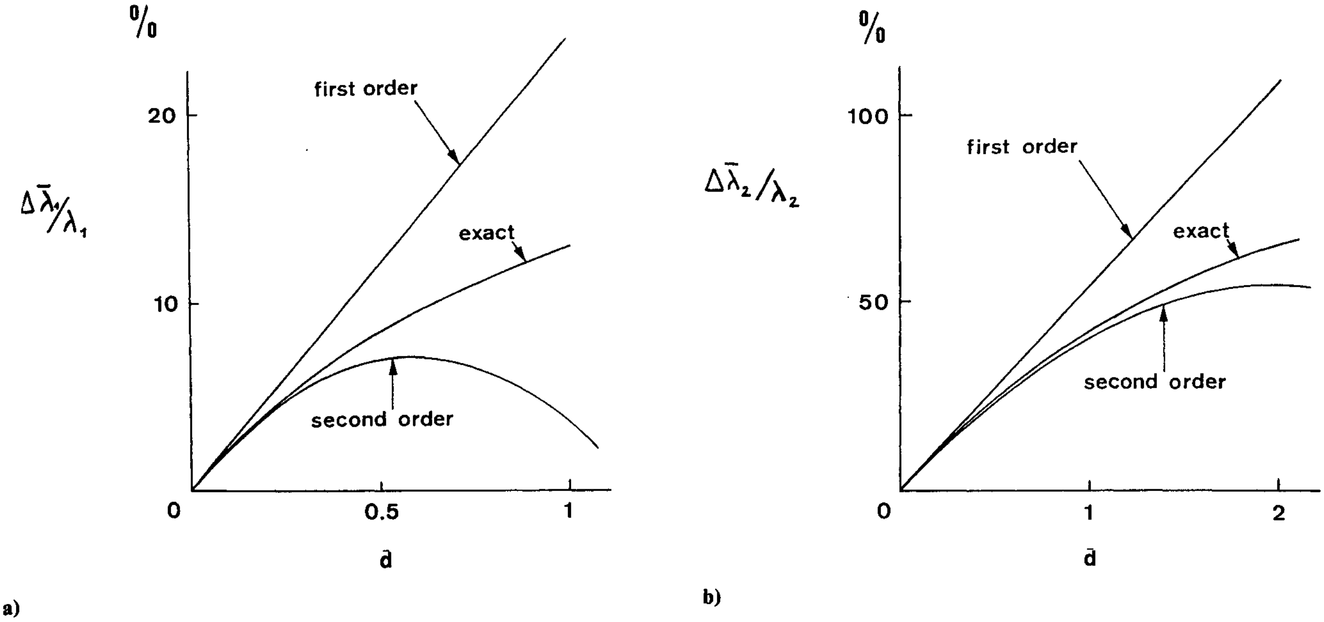

Fig. 6 First natural frequency of the discrete system defined in Fig. 5 as function of stiffness parameter $\bar{d}$.

The method proposed consists in applying the sensitivity relationships established previously to the modal model defined in Eq. (8) and constructed with N normal modes X_k when perturbation $\Delta\bar{K}$ is introduced, as shown in Fig. 3b. In most cases, we shall limit ourselves to the terms of the first order:

$$\bar{\lambda}_k = \omega_k^2 + X_{Fk}^T \Delta\bar{K} X_{Fk} \tag{28a}$$

$$\bar{X}_k = X_k - \sum_{\substack{i=1 \\ i \neq k}}^{N} \frac{X_k X_{Fk}^T \Delta\bar{K} X_{Fk}}{\omega_i^2 - \omega_k^2} \tag{28b}$$

The choice of the expression of $\Delta\bar{K}$ given in Eq. (27) is important as, in numerous cases, the matrix ΔK is not regular. For example, in the case schematized in Fig. 4 that corresponds to the introduction of a stiffener between $2n$ degrees of freedom, the matrix ΔK is or rank n and may be broken down in the following manner:

$$\Delta K = \begin{bmatrix} D & -D \\ -D & D \end{bmatrix} \tag{29}$$

It is easy to show that corrected matrix $\Delta\bar{K}$ has a similar form

$$\Delta\bar{K} = \begin{bmatrix} E & -E \\ -E & E \end{bmatrix} \tag{30}$$

In order to link matrices D and E, we partition the residual flexibility matrix in the following manner:

$$S_R = \begin{bmatrix} R_1 & Q \\ Q^T & R_2 \end{bmatrix} \tag{31}$$

By carrying Eqs. (29–31) over into Eq. (27), one obtains the following result:

$$E = [I + D(R_1 + R_2 - Q - Q^T)]^{-1} D \tag{32}$$

Thus, in the case of the introduction of a stiffener, the proposed method consists of calculating E and then applying classical sensitivity analysis software to a fictitious structure of degree N corresponding to the N modes retained. The effect of the replacement of matrix D by matrix E is to decrease the influence of the modal truncation considerably. In Ref. 25, a mathematical presentation based on using integral operators allows one to justify, generally speaking, the introduction of residual terms into the eigensolution reanalysis. In the following section, a simple example is presented in order to illustrate the efficiency of this procedure.

Example of Validation

In order to test the new procedure, we shall consider the system with three degrees of freedom shown in Fig. 5a. The perturbation consists of introducing a supplementary spring of stiffness d between points 2 and 3. The direct application of sensitivity Eq. (10) to the first two natural frequencies of the systems gives us

$$\bar{\lambda}_1^{(1)} = \lambda_1(1 + 0.2396\bar{d}) \tag{33a}$$

$$\bar{\lambda}_2^{(1)} = \lambda_2(1 + 0.5376\bar{d}) \tag{33b}$$

where

$$\bar{d} = d/d_0 \tag{33c}$$

$$d_0 = 10^{-4}\text{Nm}^{-1} \tag{33d}$$

Equation (14), which corresponds to an analysis of the second order, reveals the participation of all the higher modes. If the three modes of the system are taken into account, the following approximations are obtained:

$$\bar{\lambda}_1^{(2)} = \lambda_1[1 + 0.2396\bar{d}(1 - 0.8470\bar{d})] \tag{34a}$$

$$\bar{\lambda}_2^{(2)} = \lambda_2[1 + 0.5376\bar{d}(1 - 0.2449\bar{d})] \tag{34b}$$

Estimations $\bar{\lambda}_1^{(1)}$ and $\bar{\lambda}_1^{(2)}$ of the first eigenvalue of the perturbed system are traced as a function of stiffness parameter $\bar{d}$ in Fig. 6. The comparison with the exact eigenvalues $\bar{\lambda}_1$ shows that these estimations are only valid for low values of $\bar{d}$. If the proposed method is applied while retaining only the first mode of the system, the residual flexibility matrix is calculated by using Eqs. (18) with $L = 1$ and $N = 1$. One obtains

$$S_R = 10^{-5} \begin{bmatrix} 3.7201 & -1.4792 \\ -1.4792 & 0.9250 \end{bmatrix} \tag{35}$$

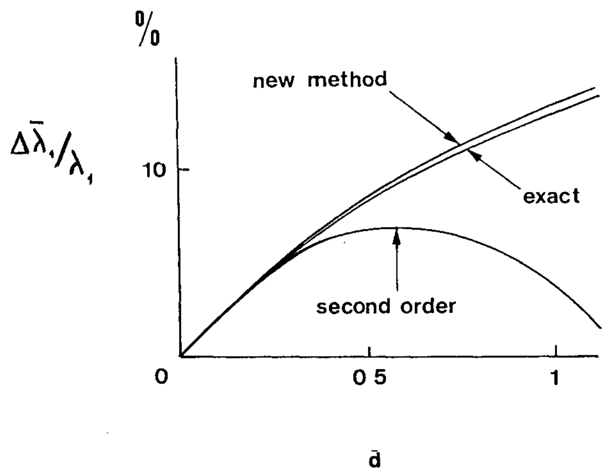

Fig. 7 First natural frequency of the discrete system defined in Fig. 5 calculated with the new method [Eq. (37)] and the classical sensitivity analysis for the second order [Eq. (34a)].

Table 1 First natural frequency of the discrete system shown in Fig. 5a with a supplementary spring of stiffness *d* between 2 and 3

Stiffness perturbation d, Nm^{-1}	Exact values Hz	Classical sensitivity analysis First order Eq. (33a)	Second order Eq. (34a)	New procedure Eq. (39a)
5000	2.687	2.730	2.668	2.690
10000	2.742	2.873	2.627	2.749

The fictitious stiffness used in the procedure is given by Eq. (32)

$$e = d(1 + 0.76035\bar{d})^{-1} \tag{36}$$

The application of the sensitivity analysis to the first order of the perturbed system by a spring of stiffness e gives us in this case

$$\bar{\lambda}_1^{(3)} = \lambda_1[1 + 0.2396\bar{d}(1 + 0.7603\bar{d})^{-1}] \tag{37}$$

As is shown in Fig. 7, this expression renders a good estimation of the exact first resonance frequency of the perturbed system. It should be noted that the classical sensitivity analysis for the second order, which was applied by taking into account all of the modes of the base structure, gives poorer results than the proposed method, which uses only the mode concerned. The results given in Table 1, which were obtained in the case $d = 5000$ Nm^{-1} and 10,000 Nm^{-1}, clearly illustrate the accuracy of the proposed method.

In agreement with the procedure, the first resonance frequency also may be estimated while retaining two modes of the base system ($N = 2$). The calculation of the stiffness equivalent e must use a lower residual flexibility matrix that, in this case, corresponds to the static participation of the third mode,

$$S_R = 10^{-5}\begin{bmatrix} 1.8656 & -0.1734 \\ -0.1734 & 0.0161 \end{bmatrix} \tag{38}$$

It should be noted that in this case the matrix S_R is not regular. The sensitivity analysis performed on the modal model with two degrees of freedom, perturbed by stiffness e, gives at the first order

$$\bar{\lambda}_1^{(4)} = \lambda_1[1 + 0.2396\bar{d}(1 + 0.2229\bar{d})^{-1}] \tag{39a}$$

and at the second order

$$\bar{\lambda}_1^{(5)} = \lambda_1[1 + 0.2396\bar{d}(1 + 0.2229\bar{d})^{-1}] \times [1 - 0.6197\bar{d}(1 + 0.2229\bar{d})^{-1}] \tag{39b}$$

In Fig. 8, these approximate values are compared to the exact values. It is interesting to also compare these results with those obtained with the help of the technique proposed by Leung[19-21] to reduce the effects of modal truncation. In this manner, by using Eq. (21b) with $L = 1$ and $N = 2$, the following estimation is obtained for the first eigenvalue:

$$\bar{\lambda}_1^{(6)} = \lambda_1[1 + 0.2396\bar{d}(1 - 0.8409\bar{d})] \tag{40}$$

This expression is found to be quite close to that given in Eq. (34a), which takes into account the participation of all the base system modes. Nevertheless, as is shown in Fig. 8, this estimation is clearly less accurate than that obtained at the same order through the use of the proposed method. It is also interesting to compare the modification of the mode shapes obtained at the first order through the use of the two truncation methods. Hence, in Table 2, the shapes of the first mode calculated by using the proposed method and Eq. (21a) with $L = 1$ are compared to the exact mode shape in the case $d = 5000$Nm^{-1}

In the case of $N = 2$, the second natural frequency may be estimated. Thus, in Table 3 the diverse approximations for the case $d = 2000$ and 5000 Nm^{-1} have been reported.

In the case of a sufficiently isolated frequency ω_i, it is possible to define a residual flexibility S_{Ri} that takes the lower and higher modes into account. Hence, for the i mode, an estima-

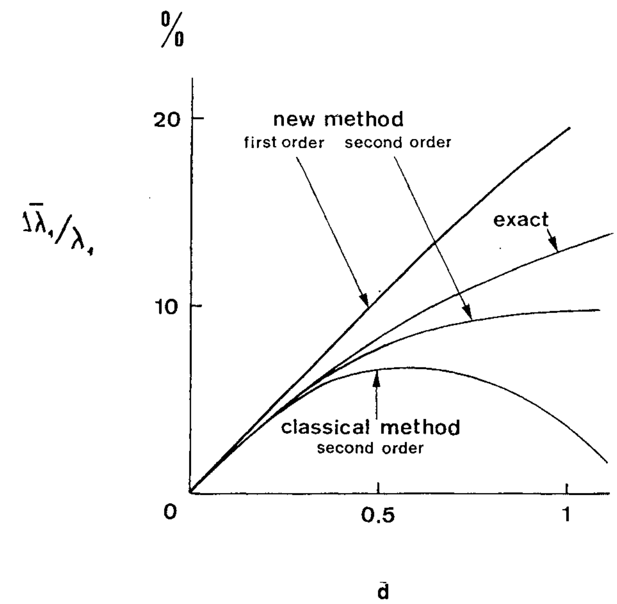

Fig. 8 First natural frequency of the discrete system defined in Fig. 5 calculated with the new method [Eqs. (39a) and (39b)] and the classical sensitivity analysis for the second order with modal truncation [Eq. (34b)].

Table 2 First mode shape of the discrete system shown in Fig. 5a with a supplementary spring of stiffness $d = 5000$ Nm^{-1} between points 2 and 3

No. of nodes	Exact values	Sensitivity analysis Classical Eq. (21a)	New procedure Eq. (28b), $N = 2$
1	0.1356	0.1408	0.1441
2	0.1879	0.1949	0.1866
3	0.2434	0.2406	0.2415

Table 3 Second natural frequency of the discrete system shown in Fig. 5a with a supplementary spring of stiffness *d* between nodes 2 and 3

Stiffness perturbation d, Nm^{-1}	Exact values Hz	Classical sensitivity analysis: First order Eq. (10)	Classical sensitivity analysis: Second order Eq. (21b) $L=1$ $N=2$	New procedure Eq. (28b) $N=2$
2000	7.513	7.530	7.516	7.514
5000	7.958	8.059	7.979	7.973

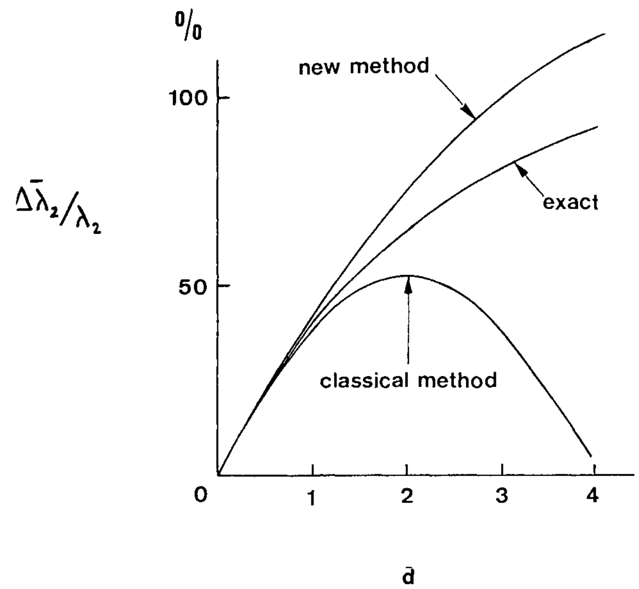

Fig. 9 Second natural frequency of the discrete system defined in Fig. 5 calculated with the new method [Eq. (42)] and the classical sensitivity analysis for the second order [Eq. (34b)].

tion of its resonance frequency may be calculated by using a stiffness matrix perturbation $\Delta\bar{K}$ obtained with the following residual flexibility matrix:

$$S_{Ri} = S_T - \sum_{k=1}^{i} \frac{X_k X_k^T}{\omega_k^2} - \sum_{k=1}^{i-1} \frac{X_k X_k^T}{\omega_i^2} \qquad (41)$$

This approach applied to the second mode of the test system defined in Fig. 4 gives the following estimation:

$$\bar{\lambda}_2^{(3)} = \lambda_2[1 + 0.5376\bar{d}(1 + 0.1917\bar{d})^{-1}] \qquad (42)$$

Although this procedure reduces the analysis to a system with a single degree of freedom, better results are obtained than with the help of a sensitivity analysis at the direct second order using all of the system's modes, as is shown in Fig. 9.

Extension of the Method

The proposed method was presented in the case of a stiffness perturbation. However, it is possible to extend it to a more complex modification of the following form:

$$\Delta\tilde{K} = \Delta K - \omega^2 \Delta M \qquad (43)$$

The correction of this matrix, introduced so as to reduce the effects of modal truncation, is always governed by Eq. (27)

$$\Delta\bar{K} = [I - (\Delta K - \omega^2\Delta M)S_R]^{-1}(\Delta K - \omega^2 \Delta M) \qquad (44)$$

The residual flexibility matrix S_R corresponds to an expansion to the first order of the participation of the nonretained modes [Eq. (25)] but also may take into account the influence of the lower modes as the preceding example has shown. As the norm of matrix $(\Delta K - \omega^2\Delta M)S_R$ is small before 1, $\Delta\bar{K}$ may be approximated by the following relation:

$$\Delta\bar{K} = P_1 - \omega^2 P_2 + \omega^4 P_3 \qquad (45)$$

where

$$P_1 = \Delta K - \Delta K S_R \Delta K \qquad (46a)$$

$$P_2 = \Delta M - (\Delta M S_R \Delta K + \Delta K S_R \Delta M) \qquad (46b)$$

$$P_3 = \Delta M S_R \Delta M \qquad (46c)$$

The modes of the perturbed system are solutions of an eigenvalue problem of a higher order than that of the base system given in Eq. (8), that is,

$$(\Omega^2 + X_F^T P_1 X_F)Q - \omega^2(I + X_F^T P_2 X_F)Q - \omega^4 X_F^T P_3 X_F Q = 0 \qquad (47)$$

In classical fashion, this problem may be reduced to a standard form

$$(H + \Delta H)Y = \omega^2(R + \Delta R)Y \qquad (48)$$

where

$$H = \begin{bmatrix} 0 & \Omega^2 \\ \Omega^2 & -I \end{bmatrix} \qquad (49a)$$

$$R = \begin{bmatrix} \Omega^2 & 0 \\ 0 & 0 \end{bmatrix} \qquad (49b)$$

$$\Delta H = \begin{bmatrix} 0 & X_F^T P_1 X_F \\ X_F^T P_1 X_F & -X_F^T P_2 X_F \end{bmatrix} \qquad (49c)$$

$$\Delta R = \begin{bmatrix} X_F^T P_1 X_F & 0 \\ 0 & X_F^T P_3 X_F \end{bmatrix} \qquad (49d)$$

The eigenvalues in the nonperturbed system correspond to angular frequencies ω_i and the eigenvectors are of the following form:

$$Y = \begin{Bmatrix} Q \\ \omega^2 Q \end{Bmatrix} \qquad (50)$$

The proposed method consists in applying the sensitivity analysis to this system by using in the calculations only the N modes retained. By limiting oneself to a development in the first order, the eigenvalues of the perturbed system are obtained through the use of a relation analogous to that given in Eq. (10):

$$\bar{\lambda}_k = \bar{\omega}_k^2 = \omega_k^2 + Y_k^T(\Delta H - \omega_k^2 \Delta R)Y_k(Y_k^T R Y_k)^{-1} \qquad (51)$$

Table 4 First natural frequency of the discrete system shown in Fig. 5b for different values of mass perturbation Δm_3 and stiffness K_4

Mass perturbation m_3, kg	Stiffness perturbation K_4, Nm^{-1}	Exact values Hz	Classical sensitivity analysis: First order Eq. (10)	Classical sensitivity analysis: Second order Eq. (14) $N=3$	New procedure Eq. (52)
2	2000	2.665	2.705	2.648	2.677
2	5000	2.926	3.094	2.879	2.963
5	5000	2.688	2.882	2.494	2.715

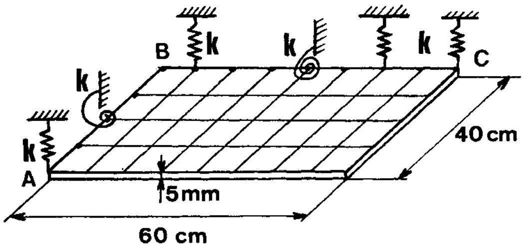

Fig. 10 Plate finite-element model.

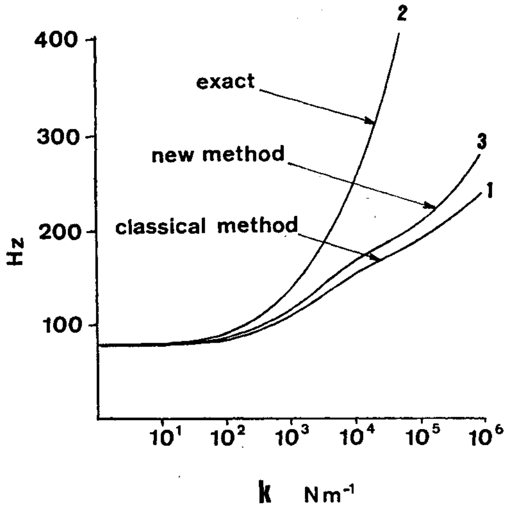

Fig. 11 Natural frequency of the second elastic mode of the free-free plate as function of stiffness k.

This equation takes into account the fact that vectors Y_k are not orthonormal as in Eq. (12). By using Eqs. (46) and (49), one ultimately obtains

$$\bar{\omega}_k^2 = \omega_k^2 + X_{Fk}^T\{\Delta K - \Delta K S_R \Delta K - \omega_k^2[\Delta M - (\Delta M S_R \Delta K + \Delta K S_R \Delta M) + \omega_k^2 \Delta M S_R \Delta M]\}X_{Fk} \quad (52)$$

To test the efficiency of this relation, we have increased mass m_3 by a perturbation Δm_3 and have added a spring of stiffness K_4 in the test model as shown in Fig. 5b. In Table 4, the first natural frequency is the perturbed system calculated through the use of Eq. (52) for several values of Δm_3 and K_4 are compared to those obtained by applying a direct sensitivity analysis to the first and second orders when all of the system's modes are taken into account.

Conclusion

The method that has been proposed allows the reduction of the effects of modal truncation during eigensensitivity analysis. The method is based on a correction of the perturbation terms before application of classical sensitivity formulas. This procedure was validated in this presentation by a system of three degrees of freedom. However, it has been applied successfully in the case of structures discretized by the finite-element method.[26] For instance, the evolution of the frequency of the second elastic mode of the rectangular free-free plate shown in Fig. 10 was analyzed when identical translational and rotational stiffnesses introduced to clamp the nodes spread along sides AB and BC. The finite-element model of the plate used contained 135 degrees of freedom. Curve 2 in Fig. 11 corresponds to the application of the sensitivity relationship at the second order [Eq. (14)] when nine modes of the base structure are used. In the light of the example treated, it thus apears that the method proposed, which may easily be programmed on a microprocessor, clearly improves the results obtained through eigensolution reanalysis.

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